Ludics and Anti-Realism

Alain Lecomte[*]
Myriam Quatrini[†]
Marie-Renée Fleury[‡]

1 Introduction

In this chapter we try to give a flavour of Ludics, a frame developed by Jean-Yves Girard on the basis of Linear Logic (cf [8, 9, 10]). Ludics seems to be very on purpose in a book devoted to logics and anti-realism because it makes no assumptions on the existence of the external world, in the sense that it does not require any "model-theoretic" assumption (like the evidence of a concept of "Truth") in order to enjoy good properties, like a special form of completeness.

On the technical side, it may be seen as a new kind of "semantics" for computer science : proofs (or *designs* as we shall see later on) may be seen as interacting processes, a view which is strongly relevant in today technology : computers do not have access to an external reality by means of a relation like "denotation", and this remark may be extended to the case of our minds which do not either access to such a "reality" by some direct relation, like it is wrongly assumed in denotational semantics.

On the philosophical side, it is the first radical attack against the traditional dualism which opposes the syntactic aspect of logics (the "language") to the denotational one ("the world"), or in other words : proof theory to model theory. One of the most famous claims made by Girard is his slogan according to which "the meaning of rules is inside the rules themselves". Of course, said like that, it seems very elliptic. In fact what Girard puts in evidence is the geometrical structure underlying logic, so that the meaning of rules is "to be found in the well hidden geometrical structure of the rules themselves".

Some of the main geometrical properties a system can have are symmetry and orthogonality. To begin, we shall refer mainly to the first one because it happens that it plays an

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[*]UMR "Structures Formelles de la Langue", CNRS-Université Paris 8 - Vincennes-Saint-Denis
[†]UMR "Institut de Mathématiques de Luminy", CNRS-Aix-Marseille Université
[‡]UMR "Institut de Mathématiques de Luminy", CNRS-Aix-Marseille Université

obvious role in logic. To say more, it seems that what makes logic to introduce deep insights in other fields, like computer science, biology or linguistics relies on its dynamics (classically expressed by the cut-elimination procedure), which is made possible because of this geometrical property. In the cut-elimination procedure, rules interact with others, introduction rules on the left with introduction rules on the right and the cut-rule itself may be viewed as a symmetric rule with regards to the identity axiom.

A well established way to consider the symmetries in logic relies on the metaphor of *games*. In games, symmetric positions are occupied by two players (the proponent and the opponent) and rules correspond to each others simply by interchanging the roles of the players. This had already been well known since seminal ideas expressed by Gentzen (as a metaphor) and then works by Lorentz [14], Lorenzen[15], Felscher[7], Hintikka [12] etc. but what is new in Girard's approach is the fact that he does not want to start from a pre-existing logic, trying to give it a *game semantics*, he proposes to look abstractly for "games" having good geometrical properties, in order to recover a corresponding logic. In any way, this is not semantics : semantics and syntax (proofs) are supposed to meet. In order to achieve such a program, we have to abstract over syntax in the same time that we have to concretize semantics, that means that we can no longer work with a "bureaucratic" management of symbols which consists in establishing a one-to-one correspondance between symbols on one side and alleged "real" entities on the other. The structures we shall obtain (*designs*, see below) will keep, as their syntactic part, a kind of skeleton of a *proof*, and as their semantic one, an alternation of steps (negative and positive) which makes them immediately interpretable as *strategies*.

Moreover, such an approach amounts to take geometrical notions very seriously : there are only locations, or *loci* ! Such a view is very promising for several fields of thought, not only philosophy but also linguistics, or more precisely : *pragmatics*. Much has been said for instance on the inability of traditional logic to handle the fact that in discourse, utterances, even if expressed by the same *type*, stay different as *tokens* (see Hamblin [11]). Ludics grasps the *token* reality of language.

As it seems obvious, such a move is a radical change with regards to the Tarskian tradition which is based on a primitive notion of Truth, formalized in Model Theory, and for which the ultimate foundation of validity lies outside logics. In Ludics, there is no outside ! There are only processes which interact : we shall say that they enter normalization processes, which may converge or diverge. Truth is replaced by a notion of *winning strategy*, that is of a design which wins against any other one, and, formulae, which were already seen as sets of their proofs in intuitionnistic logic are now replaced by "behaviors", that is sets of strategies satisfying some completeness property. In this sense we can say that it provides a pure anti-realist (or "internal") conception to logics.

In the following, we shall study in the first section the origins and the methodology of Lu-

dics, insisting on anterior works (like Andréoli's) which put the evidence on the polarity of logical beings. Then in the second section, we shall study designs as ways of representing proofs in a more abstract way, and we shall make the link with strategies in a game ("designs") by means of some small (trivial) examples. In the third section, we shall reconstruct notions of traditional logic on this new base. Exactly like in intuitionistic logic, formulae are interpreted as sets of proofs, here formulae may considered as sets of designs, or behaviors. Finally we recover some flavour of "truth" and "falsehood", but not from a metaphysical (or *essentialist* as Girard currently says) viewpoint, only from a technical one, linked to the existence of strategies. More importantly, we look at how to recover the basic connectives from behaviors, thus giving a new interpretation to the connectives of Linear Logic.

2 The origins of Ludics

2.1 Monism in Logics

Ludics comes from a reflection by Jean-Yves Girard on the deep meaning of logics. Traditionally, logicians live in a dualist universe : on one side there is syntax, axiomatic systems and so on, on the other side, semantics, that is, mainly, model theory. The situation has been so at least since the time of Tarski. Van Heijenhoort, who inspired Hintikka ([13]), highlighted the opposition between two trends in the history of modern logics, one, mainly represented by Frege, the first Russell and the Vienna Circle in his syntactic period, for whom logic is a universal language (it has no outside), the other one, represented by Shröder, Löwenheim, Gödel and Tarski for which we may on the contrary make the interpretations to vary by means of model theory, thus relativizing the scope of logics and giving it the characteristic of a calculus. The second trend seems to have been overwhelming during the biggest part of the Twentieth Century.

In this traditional trend, a formula, in order to be validated, must follow from a *deduction*, but in order to be invalidated, a *counter-model* is given. In this way, *proofs* are opposed to *counter-models*. This conception leads logicians to a constant jump between two worlds : SYNTAX and SEMANTICS.

Since his paper *On the meaning of logical rules* ([8]), Girard has tried to abolish this dualist conception by replacing it by a monist one, in terms of which proofs no longer oppose to counter-models, but to *counter-proofs* (or, if we prefer, "refutations"), that is objects of the very same nature. Such an enterprise looks very much like Game Theory since the latter is based on the confrontation between two players (the proponent and the opponent), such that one of them proposes a formula and defends it against the attacks of the other until one of the two players has no longer moves to play.

Game Semantics has already been proposed as a semantics for Linear Logic ([3, 1]), and it is known that connectives of Linear Logic are well interpreted in this frame. The originality of Ludics will consist in its defining objects only starting from their behavior with regards to each other, by means of a particular notion of orthogonality : two objects are orthogonal when they enter a normalizing process which converges. In standard presentations of logics (sequent calculus and proof-nets), this dynamical aspect shows up in the process of cut-elimination. Even though, in ludics, the cut-rule is not expressed by an explicit rule, it is, as we shall see, ubiquitous : it resides in the mere co-presence of two objects at the same locus. Its elimination will therefore represent the dynamical aspect of the system.

In the Game Semantics approaches, the meaning of logical rules is already inside these rules themselves, for instance it resides in their symmetries. We can give *negation* as an illustration, which must not be interpreted as NO (by a single inversion of truth-values) but by the exchange of the proponent and the opponent positions. Girard goes even further by assuming that this is not a mere "description" of an operation already given, but its constitution itself (as pointed out by A. Pietarinen [16] who quotes Wittgenstein on that aspect).

The position held by Girard is therefore *immanentist* or *anti-realist*, in complete opposition with the tarskian conception which leads to a continual escape towards *meta*-levels when it tries to give foundation to the concept of truth, a conception that can be seen by contrast to the transcendantal one which is implied by this tarskian view and the need to believe in the meta-levels if we want a foundation for the semantics of formulae.

With this new perspective, the logician may live in a homogeneous space. The embryon of such a view may be seen in Heyting's semantics of proofs, which can lead to the following observations concerning the notion of *test*.

1. how to test $(A \wedge B)$? by testing A or by testing B,

2. how to test $(A \vee B)$? by testing A and by testing B

3. how to test $(A \Rightarrow B)$? by admitting A and testing B

The notion of *test* appears to be close to that of *negation* (cf. De Morgan rules) and we are led to the analogy : *test for A = proof of $\neg A$*. But of course, if A is provable, such a proof cannot exist ! It is not possible to have altogether a proof of A and a proof of $\neg A$. A test for A is therefore what Girard calls a *paraproof*. Hence the fact that, from now on, paraproofs will be opposed to other paraproofs, only some of them becoming true proofs : we of course don't know which of them before making the interaction to begin.

The intuitionistic viewpoint is not the best one for developing those ideas since negation is not involutive in it (the fact that $\neg A$ would not pass the test would not entail the validity of A). Girard says that it is not that intuitionism would define a *bad* negation, but because of

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its geometrical limitations : since the only correct meaning of negation is the exchange, this requires the two sides of a sequent be symmetric. On the other side, the classical viewpoint is embarrassing because of its non-constructive feature : the normalization procedure is not confluent. Hence the idea of symetrizing the intuitionistic viewpoint and going to Linear Logic.

Girard's idea may be seen as an attempt to maximally abstract over syntax (in particular to get rid of the "bureaucratic" management of symbols) and concretizing semantics, both moves meeting in a middle term where it remains a mere geometrical object, which is a game. We shall then meet a notion of completeness, but of course not the same as the tarskian one. This notion was already there in Game Semantics.

> *if σ is a winning strategy for the game $|A|$, then there exists a proof π of A such that $\sigma=|\pi|$*

where $|A|$ is the game which interprets the formula A and $|\pi|$ the strategy which interprets the proof π.

This is reminiscent of many efforts made during the twentieth century to found Logics on Game Theory ([14], [15]), but for Girard, these attempts were *ad hoc* : they all started from the existing logic and then tried to establish good game rules in accordance with deduction rules in order to assimilate provable formulae and formulae for which there exists a winning strategy. Girard is looking for a notion (still of a "game" ?) which would be simpler, more natural and "geometrical" (so that for instance it integrates the dynamical notion of cut-elimination), and, for him, only then, we would be able to determine the suitable logic. In order to fully characterize an object (like a proof) by means of its interactions with other ones, there must be enough objects with which to interact, that is also objects which are... not necessarily proofs ! Another way of drawing a similar conclusion consists in asking : if proofs give winning strategies, what is there concerning the non winning ones ? In both questions, the answer is : *paraproofs*, that is... *false* proofs ! However, if we can build false proofs, that necessarily comes from that we accept *false* rules, that is : *paralogisms*. A new notion of completeness arises, which as claimed by Girard, is much more interesting than the previous one : a notion of *internal* completeness, according to which a set is said to be *complete* if it contains all the objects needed in order to be closed by interaction.

Not any paralogism is accepted and only the least. Moreover they will have to conform to the cut-elimination procedure too, thus limiting the kind of paralogisms that can exist.

The main paralogism Girard proposes consists in considering any sequent $\vdash \Gamma$ as an axiom. It is what happens when we try to make a proof in the sequent calculus : at a certain point of the proof, we arrive at a sequent we can see that we shall not be able to prove it. At this point, we give up, but the object we met was a paraproof, it can be distinguished by the application of this axiom which may be understood as : "I give up". This rule is called

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the *daimon*. [1] We can see that, at this stage, "the" logic is not yet determined. Perhaps we shall find a known logic (intuitionistic ? classical ? linear ? affine ?) but it is not sure. Even though Girard would be encline to choose linear logic, he does not exclude other logics like the affine one, or even several logics. The only remaining criterion for a "good" logic is its behaviour with regards to cut-elimination.

2.2   Methodological considerations

If we want to interpret a proof as a strategy in a game with two players, then necessarily must intervene somewhere the notion of polarity, which exists between the two players. J. M. Andreoli ([2]) had already met this notion in his analyses of the proofs in Linear Logic. He discovered that proof-search is more tractable if we content ourselves only with proofs of a certain type, which form a complete set with regards to the provable sequents. Actually, the connectives of Linear Logic may be split in two groups which behave differently with regards to the active formula, at each step.

- the negative connectives :
    - the multiplicatives $\perp$, $\wp$ and ?
    - the additives : T, $\&$, $\forall$
- the positive connectives :
    - the multiplicatives : 1, $\otimes$ et !
    - the additives : 0, $\oplus$, $\exists$

Negative connectives are deterministic and reversible. In effect, when the [$\wp$] or the [$\&$]-rule is applied, there is only one way to do it, and consequently, it is always possible to go back : no information is lost when the rule is applied. But when the [$\oplus$] or the [$\otimes$]-rule is applied, there are many ways to continue the proof (always seeing the proof from

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[1]There could be other paralogisms. Let us consider for instance a player I (Girard, p. 13) who tries to prove $\vdash ?A \otimes ?B$. Of course, this sequent is not valid if isolated, but it could happen that for certain A and B it is provable. Then the opponent should try to prove $\vdash !A^{\perp} \wp !B^{\perp}$, which amounts to prove $\vdash !A^{\perp}, !B^{\perp}$ which is of course not provable in standard Linear Logic. If the player II has only the *daimon*-rule to oppose, then I easily wins, but perhaps too easily because the immediate abandon of II prevents from entering into the formula $\vdash ?A \otimes ?B$ (and understand in the present case why it is valid). Therefore we have perhaps to add other paralogisms. For instance, if we want to decompose the formula, we can be authorized to use weakening. In this case, the player II would play $\vdash !A^{\perp}$ or $\vdash !B^{\perp}$. Other solution : (s)he could use the Mix-rule and try to prove the two sequents $\vdash !A^{\perp}$ **and** $\vdash !B^{\perp}$, where the mix-rule is the following :

$$\frac{\Gamma \vdash \Delta \quad \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda} \, Mix$$

(It is not accepted in classical Linear Logic whereas it is satisfied in numerous models of Linear Logic, like coherent spaces)

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bottom to the top). In case of $A \oplus B$, it is necessary to choose between continuing on A and continuing on B, in case of the $[\otimes]$-rule, splitting the context is not deterministic. This entails that it will be difficult to reverse the process when a choice has been made : in the first case, B will be forgotten if A has been chosen. in the second case, if we reverse the rule, we obtain a false rule since any partition of the context does not fit to the expected conclusion.

Hence the idea according to which an efficient proof-search must select at each step the main formula so that :

- if the sequent contains negative formulae, then any of them may be chosen (at random) and the proof may be continued in this way until there is no more negative formula in the sequent to prove (this phase is deterministic)
- when all the negative formulae have been decomposed, a main formula must be selected in a non-deterministic manner, but as soon as it has been selected, it is necessary to focus on it in such a way that in the following, only subformulae of this formula are selected, as long as the subformula is positive.

It appears that *it is possible to present a proof under the form of an alternate sequence of steps* : a sequence of negative steps followed by a sequence of positive ones, followed again by a sequence of negative steps and so on. Even better : we may consider a sequence of steps of the same sign as a single step of that sign, consisting in applying a rule associated with a synthetic connective.

With such connectives, and for a proof consisting in such a sequence of alternate steps, the focalization method preserves as an invariant the fact that a sequent contains at most one negative formula (all the negative connectives have been grouped together in the same formula). If the active formula is positive, then all the other formulae of the sequent are positive (if not, the negative one would have been chosen as the active formula) and all the premisses in the application of the rule have exactly one negative formula, each of which arises from the decomposition of the active formula. If the active formula is negative, all the other formulae are positive and each premisse is a sequent consisting only in positive formulae. At the beginning, we want to prove a sequent which contains a single formula, which is conform to the invariant.

To consider the polarized proofs is particularly interesting for the analysis of the objects which prevails in the development of Ludics and which can be summarized in the following slogan : "keep of the proof object only what is concerned by cut elimination". The relevant equivalences between proofs from the proof search point of view are also relevant from the cut elimination one. The quotient of the polarized proofs realized by using synthetic connectors is thus relevant to approximate good objects and it is exactly what Ludics will do. Let us illustrate this matter by means of an example.

Let us consider the following equivalent LL-formulae : $(A{\otimes}C){\oplus}(B{\otimes}C)$ and $(A{\oplus}B){\otimes}C$

and let us suppose we are eliminating a cut on such a formula (by putting them in interaction with $(A^\perp \wp C^\perp)\&(B^\perp \wp C^\perp)$ or $(A^\perp \& B^\perp)\wp C^\perp$ for instance). The proof above $(A \otimes C) \oplus (B \otimes C)$ can be :

$$either \quad 1 = \cfrac{\cfrac{\vdash A \quad \vdash C}{\vdash A \otimes C}}{\vdash (A \otimes C) \oplus (B \otimes C)} \qquad or \qquad 2 = \cfrac{\cfrac{\vdash B \quad \vdash C}{\vdash B \otimes C}}{\vdash (A \otimes C) \oplus (B \otimes C)}$$

The continuation of the cut elimination procedure leads to :
  – eliminate one cut on $\vdash A$ and one cut on $\vdash C$ if the proof is 1
  – eliminate one cut on $\vdash B$ and one cut on $\vdash C$ if the proof is 2
If we consider the other formula, namely $(A \oplus B) \otimes C$, again there are two possible proofs above this formula :

$$either \quad 1' = \cfrac{\cfrac{\vdash A}{\vdash A \oplus B} \quad \vdash C}{\vdash (A \oplus B) \otimes C} \qquad or \qquad 2' = \cfrac{\cfrac{\vdash B}{\vdash A \oplus B} \quad \vdash C}{\vdash (A \oplus B) \otimes C}$$

Once again there are two possibilities :
  – if the proof is $1'$, continue on $\vdash A$ and $\vdash C$;
  – if the proof is $2'$, continue on $\vdash B$ and $\vdash C$.
The proofs 1 and $1'$ on the one hand, 2 and $2'$ on the other hand are indistinguishable according to the cut elimination procedure. Because ludics only preserve the information "to continue on A and C" or "to continue on B and C", it will make no difference between 1 and 1' on one side and 2 and 2' on the other.
The following example taken from [4] will make clear the role of synthetic connectives in displaying proofs in sequences of alternate steps.
Let us consider the formula : $((N_1 \otimes N_2)\&Q)\wp R$ where Q, R are supposed to be positive, $P = N_1 \otimes N_2$ is positive and N_1, N_2 are negative. $(P\&Q)\wp R$ is negative. The rule for the synthetic connective (a synthese of $\&$ and $\wp$) is :

$$\cfrac{\vdash P, R, \Lambda \qquad \vdash Q, R, \Lambda}{\vdash (P\&Q)\wp R, \Lambda}$$

A focalized proof of $((N_1 \otimes N_2)\&Q)\wp R$ ends up like this :

$$\frac{\dfrac{\vdash N_1, R, \Lambda_1 \qquad \vdash N_2, \Lambda_2}{\vdash (N_1 \otimes N_2), R, \Lambda \qquad \vdash Q, R, \Lambda}}{\vdash ((N_1 \otimes N_2) \& Q) \wp R, \Lambda}$$

where we can see the alternation of steps : first a negative rule, then a positive one (here the tensor rule). The same proof may be presented in the following way :

$$\frac{\dfrac{N_1^\perp \vdash R, \Lambda_1 \qquad N_2^\perp \vdash \Lambda_2}{\vdash (N_1 \otimes N_2), R, \Lambda \qquad \vdash Q, R, \Lambda}}{((N_1 \otimes N_2)^\perp \oplus Q^\perp) \otimes R^\perp \vdash \Lambda}$$

In this presentation, there are only positive connectives. Moreover, there is at most one formula on the left of $\vdash$.

All these observations lead us to simplifying the presentation of the calculus : we shall consider only sequents with only positive formulae and at most one formula on the left hand side. We have therefore only positive rules, but for each synthetic connective, we have a left introduction rule, which is reversible, and a set of right introduction rules which are not reversible, for instance for the synthetic connective above, which may be written as $(P^\perp \oplus Q^\perp) \otimes R^\perp$:

$$\frac{\vdash P, R, \Lambda \qquad \vdash Q, R, \Lambda}{(P^\perp \oplus Q^\perp) \otimes R^\perp \vdash \Lambda} \{\{P, R\}, \{Q, R\}\} \qquad\qquad \frac{P \vdash \Gamma \qquad R \vdash \Delta}{\vdash (P^\perp \oplus Q^\perp) \otimes R^\perp, \Gamma, \Delta} \{P, R\}$$

$$\frac{Q \vdash \Gamma \qquad R \vdash \Delta}{\vdash (P^\perp \oplus Q^\perp) \otimes R^\perp, \Gamma, \Delta} \{Q, R\}$$

the rule on the top right comes from the following deduction :

$$\frac{\dfrac{\dfrac{P \vdash \Gamma}{\vdash P^\perp, \Gamma}}{\vdash P^\perp \oplus Q^\perp \Gamma} \quad \dfrac{R \vdash \Delta}{\vdash R^\perp \Delta}}{\vdash (P^\perp \oplus Q^\perp) \otimes R^\perp, \Gamma, \Delta}$$

We may notice (cf. Curien, p.17) that the connective isomorphic to it : $(P^\perp \otimes Q^\perp) \oplus (P^\perp \otimes R^\perp)$ has exactly the same rules.

3 Designs

3.1 Proof trees which may be infinite

If now, we would like to write all the rules for all the synthetic connectives, that would make an infinity of rules. We therefore must content ourselves with rule schemata. And because we cannot foresee in such schemata what will be the particular formulae, we shall content ourselves with locations (*loci*), that is, the place where they are inscribed.

As a matter of example, let us take again the three rules that we stated above, concerning the synthetic connective $(P^\perp \oplus Q^\perp) \otimes R^\perp$, and that we recall here :

$$\frac{\vdash P, R, \Lambda \qquad \vdash Q, R, \Lambda}{(P^\perp \oplus Q^\perp) \otimes R^\perp \vdash \Lambda} \{\{P, R\}, \{Q, R\}\} \qquad\qquad \frac{P \vdash \Gamma \qquad R \vdash \Delta}{\vdash (P^\perp \oplus Q^\perp) \otimes R^\perp, \Gamma, \Delta} \{P, R\}$$

$$\frac{Q \vdash \Gamma \qquad R \vdash \Delta}{\vdash (P^\perp \oplus Q^\perp) \otimes R^\perp, \Gamma, \Delta} \{Q, R\}$$

Let us assume that the atomic formulae are stored at various addresses, these ones being words on $\mathbb{N}$ (the set of integers), and that $P^\perp, Q^\perp, R^\perp$ are located at the respective addresses 1, 2, 3. Let ξ be a variable of address. $\xi \star I$ denotes the set $\{\xi i; i \in I\}$ and ξi is often noted $\xi \star i$. Then, if the formula $(P^\perp \oplus Q^\perp) \otimes R^\perp$ is supposed to be stored at the address ξ, the atomic formulae P, Q, R which occur in the premisses are supposed to be stored at the respective addresses $\xi 1, \xi 2$ and $\xi 3$. $1, 2$ and 3 thus occur as relative addresses. Sequents are normalized, in the sense that they are written down under the form of what Girard calls a *fork*, with at most one address on the left hand side. The two types of sequents are therefore : $\xi \vdash \Lambda$ and $\vdash \Lambda$. The proofs of sequents of the first (resp. second) type are said to have a negative (resp. positive) base.

Thus the first rule on top left is a negative base rule (the positive connective is introduced on the left), whereas the two others are positive rules (the positive connective is introduced on the right). A negative rule is therefore a rule with several sequents as premisses, each one containing several addresses on their right (and nothing on their left), these sequents are of the form $\vdash \xi \star J, \Lambda_J$. A positive rule is a rule with several premisses, each one having one address on its left. Those premisses are therefore of the $\xi \star i \vdash \Lambda_i$ form.

The three rules above become :

$$\frac{\vdash \xi_1, \xi_3, \Lambda \qquad \vdash \xi_2, \xi_3, \Lambda}{\xi \vdash \Lambda} (-, \xi\{\{1, 3\}, \{2, 3\}\}) \qquad\qquad \frac{\xi_1 \vdash \Gamma \qquad \xi_3 \vdash \Delta}{\vdash \xi, \Gamma, \Delta} (+, \xi, \{1, 3\})$$

$$\frac{\xi_2 \vdash \Gamma \qquad \xi_3 \vdash \Delta}{\vdash \xi, \Gamma, \Delta} (+, \xi, \{2, 3\})$$

and the general schemata of the rules are the following :

POSITIVE RULE

$$\frac{\dots \quad \xi \star i \vdash \Lambda_i \quad \dots}{\vdash \xi, \Lambda} \, (+, \xi, I)$$

NEGATIVE RULE

$$\frac{\dots \quad \vdash \xi \star J, \Lambda_J \quad \dots}{\xi \vdash \Lambda} \, (-, \xi, \mathcal{N})$$

where I and J are finite subsets of $\mathbb{N}$, $i \in I$, with the Λ_i pairwise disjoints, $\mathcal{N}$ is a set (*possibly infinite*) of finite subsets of $\mathbb{N}$, each J of the negative rule being an element of this set, all the Λ_J are included in Λ, and moreover each base sequent is well formed in the sense that all addresses are pairwise disjoint.

To that, we add the *daimon* :

$$\frac{}{\vdash \Lambda} \, \dagger$$

The fact that these rules don't refer to any particular formula may be glossed as saying that they are not *typed*. On this viewpoint, we understand that this system may be compared with the non typed λ-calculus.

The (para)proofs which are drawn in this system are called *designs*. A design is therefore an alternate sequence of negative and positive rules, with occasionally a *daimon* to close it, but a design may be infinite (as we shall see later on, a dispute may be endless !). In fact, there is an infinity in both dimensions : in breadth and in depth. There may be infinity In breadth also because the negative rule admits cases where there would be an infinite amount of J (infinite branching).

The general form of a design is thus the following one :

$$\cfrac{\dots \; \cfrac{\dfrac{\dots \vdash \xi_{k_{11}} \star J_2, \Lambda_{J_m} \; \dots}{\xi_{k_{11}} \vdash \Lambda_{i_1}} \, (-, \xi_{k_{11}}, \mathcal{N}_2) \quad \dfrac{\dots}{\xi_{k_{12}} \vdash \Lambda_{i_2}} \; \dots \; \dfrac{\dots}{\xi_{k_{1j}} \vdash \Lambda_{i_j}}}{\vdash \Lambda_{I_k}, \xi_{k_1}, \xi_{k_2}, \dots, \xi_{k_{n_k}}} \, (+, \xi_{k_1}, J_1) \; \dots}{\xi \vdash \Lambda} \, (-, \xi, \mathcal{N})$$

At the first step, it is assumed that $\mathcal{N} = \{I_1, \dots, I_{N_1}\}$. Only the positive sequent associated with k, that is relative to the choice of I_k in $\mathcal{N}$, has been represented as a premisse. It

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contains the elements $k_1, k_2, ..., k_{n_k}$. At the second step, we have a positive rule for each premisse of the previous step, chosen for instance in taking as the focus the first ξ_{k_l}, that is ξ_{k_1}, and that gives for each a sequence of negative premisses indexed by a set J_1, which, itself, contains the elements $1, 2, ..., j$. The Λ_{i_p} of the numerator absorb the ξ_{k_l} which have not yet been used as foci (but they will do !). Then, the design is continued by making the steps to alternate.

The positive rule may be interpreted as :
- the player selects a locus ξ to pursue the interaction, and
- from this locus, he asks a finite set of questions i

and the **negative** rule as :
- after having asked a question, the player considers a directory of possible answers $\mathcal{N}$ from his/her partner

As we have already noticed, there is no axiom rule in this system. It is replaced by *Fax*, which allows to make a correspondance between "identical" formulae occurring at different addresses (therefore not identical properly speaking, but simply "equiform"). *Fax* is the design which displays the proof of :

$$A \vdash A$$

by performing all the possible decompositions of the formula A. It has therefore the recursive definition :

$$Fax_{\xi,\xi'} = \cfrac{\cfrac{...\quad Fax_{\xi_{i1},\xi'_{i1}}\quad ...}{...\xi' \star i \vdash \xi \star i...}}{\cfrac{...\quad \vdash \xi \star J_1, \xi'\quad ...}{\xi \vdash \xi'}(-,\xi,\mathcal{P}_f(\mathbb{N}))}(+,\xi',J_1)$$

3.2   Designs as strategies

A system where objects are alternate sequences of moves with different polarities looks very much like a dialogue. Positive steps are "moves" of one participant (or player) and negative ones are those of the other player. More traditional approaches in Game Semantics (like Hintikka's school) or in dialogical logic already established a dichotomy between connectives in classical (or intuitionistic) logics. They belong to two different classes :
- the active ones : those for which the rule in dialogue consists in an active choice made by the proponent ($\vee, \exists$)
- the passive ones : those for which the associated rule consists in a passive choice, that is a choice made by the opponent ($\wedge, \forall$)

We retrieve the same dichotomy in Linear Logic : the active ones are the positive ones and the passive ones are the negative ones.

As we have seen, at a negative step, the proof displays a family of subsets of $\mathbb{N}$, $\mathcal{N}$, and at the following positive one, a $J \in \mathcal{N}$ is selected. On the dialogical view, the positive steps correspond to moves played by the proponent and the negative ones to moves played by the opponent. The opponent does not choose $\mathcal{N}$ (it is the proponent who possesses a directory of subsets for him/her), but (s)he chooses a $J \in \mathcal{N}$ (or if there is no consensus a $J \notin \mathcal{N}$).

At a positive step, the proponent chooses an address ξ and a set I ((s)he enumerates the cases (s)he is concerned with). At a negative step, the opponent chooses an address, and the proponent must be ready for the answer of the opponent. The latter may choose a J which does not belong to $\mathcal{N}$: in this case, the proponent did not anticipate on his or her opponent's move ((s)he did not choose a $\mathcal{N}$ enough big).

A *design* may then be seen as a *strategy*, and, in this case, it may be seen as a " *dessein*" (that is a *plan*). Each player possesses a function which, at every move played by his/her opponent, associates a move that (s)he can play.

Let us here take an example due to M. Quatrini.

A person X wishes to know whether another one Y, who is supposed to sell some houses and flats, will sell a goods denoted by 1. X knows that Y possesses three goods : 1, 2 and 3. X wishes to know :

1. *whether Y will send 1*

2. *in case Y would sell 1, at what price ?*

The strategy of X consists in beginning the exchange by asking the question :

0 = what goods do you wish to sell ? This is represented by a positive move, starting from an empty address, here denoted by $<>$.

$$\frac{0 \vdash}{\vdash <>}$$

starting from there, X must foresee all the possible answers by Y (therfeore the set of subsets of $\{1, 2, 3\}$) :

$$\frac{\vdash 01 \quad \vdash 01, 02 \quad \vdash 01, 03 \quad \vdash 01, 02, 03 \quad \vdash 02 \quad \vdash 02, 03 \quad \vdash 03}{0 \vdash}$$

X then foresees that for some of the answers by Y, (s)he will not pursue the dialogue, because they don't interest him or her, hence :

$$\frac{\vdash 01 \quad \vdash 01, 02 \quad \vdash 01, 03 \quad \vdash 01, 02, 03 \quad \overline{\vdash 02}^{\dagger} \quad \overline{\vdash 02, 03}^{\dagger} \quad \overline{\vdash 03}^{\dagger}}{0 \vdash}$$

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In the cases where 1 is for sale, X must plan to ask the price. For instance, if 1 is the only goods for sale, (s)he has planed to ask : *1 = at what price ?*. Hence :

$$\frac{011 \vdash}{\vdash 01}$$

In the left hand side of the sequent, there is a representation of the history of the game : X asked the question 0 and obtained the answer 1, now (s)he asks the question 1. X must be ready to receive any price i :

$$\frac{\dfrac{... \vdash 011i...}{011 \vdash}}{\vdash 01}$$

and when (s)he gets the value of i, X stops the dialogue :

$$\frac{\dfrac{... \ \dfrac{}{\vdash 011i}^{\dagger} \ ...}{011 \vdash}}{\vdash 01}$$

Because every branch is treated in this way, X's strategy (his or her design) may be represented by :

$$\frac{\dfrac{\dfrac{... \ \dfrac{}{\vdash 011i}^{\dagger} \ ...}{011 \vdash}}{\vdash 01} \quad \dfrac{\dfrac{... \ \dfrac{}{\vdash 011i,02}^{\dagger} \ ...}{011 \vdash 02}}{\vdash 01,02} \quad \dfrac{\dfrac{... \ \dfrac{}{\vdash 011i,03}^{\dagger} \ ...}{011 \vdash 03}}{\vdash 01,03} \quad \dfrac{\dfrac{... \ \dfrac{}{\vdash 011i,02,03}^{\dagger} \ ...}{011 \vdash 02,03}}{\vdash 01,02,03} \quad \dfrac{}{\vdash 02}^{\dagger} \quad \dfrac{}{\vdash 02,03}^{\dagger} \quad \dfrac{}{\vdash 03}^{\dagger}}{\dfrac{0 \vdash}{\vdash <>}}$$

Of course, on his or her side, Y has also his or her own strategy, which consists firstly in being ready to receive any question :

$$\frac{\vdash 0}{<> \vdash}$$

It is the dual move with regards to X's one :

$$\frac{0 \vdash}{\vdash <>}$$

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Then (s)he answers the question by choosing a subset of $\{1, 2, 3\}$: that amounts to applyng a positive rule.

$$\frac{\dfrac{0n \vdash \quad ... \quad 0m \vdash}{\vdash 0}}{<>\vdash}$$

At the move after, Y must be ready for receiving the question 1 and (s)he is free to give the integer i, which corresponds to some price, if desired.

A design of Y may therefore be (among others) :

$$\frac{\dfrac{\dfrac{\dfrac{011i \vdash}{\vdash 011}}{01 \vdash} \quad\quad 03 \vdash}{\vdash 0}}{<>\vdash}$$

A game (a dialogue) may be represented by a sequence of rule applications. In this case, a move is entirely determined by : a sign, an address and a subset of $\mathbb{N}$. For instance :

$$(+, <>, \{0\}), (-, 0, \{1, 3\}), (+, 01, \{1\}), (-, 011, \{i\})\dagger$$

is a game from the X viewpoint. It consists for X in :

1. first, positive, move : X chooses a question (= a set I, here $\{0\}$),
2. second, negative, move : Y chooses starting from the address 0 the subset $\{1, 3\}$,
3. third, positive, move : X chooses $\{1\}$, extending the address proposed by Y (here 01),
4. fourth, negative, move : Y, chooses a price i, extending the address 011,
5. the *daimon* stops the dialogue : X got all the informations (s)he wished.

A game may also be :

$$(+, <>, \{0\}), (-, 0, \{2, 3\}), \dagger$$

It is worth to notice that X must also foresee a design from Y which would consist in a refusal to sell anything. This design would be represented by :

$$\frac{\dfrac{\overline{\quad\quad}}{\vdash 0}^{\,\dagger}}{<>\vdash}$$

This kind of dialogue corresponds to the efforts made by a player X to demonstrate a formula ϕ, against another player Y, who tries to refute it. To take again the same example as at the end of 2.2, let us see the question of cut-elimination as a dialogue (or a *dispute*). X wants to prove $(A \otimes C) \oplus (B \otimes C)$ and Y opposes $(A^\perp \wp C^\perp) \& (B^\perp \wp C^\perp)$, that is Y negates the formula proposed by X. The design of Y is therefore :

$$\frac{\vdash A^\perp, C^\perp \quad \vdash B^\perp, C^\perp}{(A \otimes C) \oplus (B \otimes C) \vdash}$$

and the one of X may be one of the two following ones :

$$\frac{A^\perp \vdash \quad C^\perp \vdash}{\vdash (A \otimes C) \oplus (B \otimes C)} \qquad or \qquad \frac{B^\perp \vdash \quad C^\perp \vdash}{\vdash (A \otimes C) \oplus (B \otimes C)}$$

These two designs may be connected by a cut to the one of Y. By eliminating the cut (normalization), it is possible to go up to :

$$\vdash A \quad \vdash A^\perp, C^\perp \quad \vdash C$$

or :

$$\vdash B \quad \vdash B^\perp, C^\perp \quad \vdash C$$

In other words, the two players must agree on the presence of A and C or on the presence of B and C. If they agree on one of these co-presences, the interaction can continue. We see here that a game may be summarized into simpler elements into which it reduces. If another game shows up, which may be reduced in the same way, the two games are equivalent. It is exactly what happens when trying to demonstrate the equivalent formula $(A \oplus B) \otimes C$. That means that these two formulae cannot be distinguished on a normalization viewpoint.

Let us look at a more intuitive example, due to C. Faggian and F. Maurel ([6]).

Let us assume the following contract : for one euro, Bob accepts to give Alice, at her choice, either a book or, on his own choice, a CD or a DVD.

This contract takes the form of a Linear Logic formula :

$$1\ euro \multimap (1\ livre\ \&(1\ CD \oplus 1\ DVD))$$

Alice accepts the contract and has two strategies :
  – she gives one euro and she chooses the book ;
  – she gives one euro and she chooses the surprise
Bob has two strategies :

- he is ready to receive one euro and to give a book or to receive one euro and to give a CD
- he is ready to receive one euro and to give a book or to receive one euro and to give a DVD

Alice's strategies correspond to the two following beginnings of proofs :

$$\dfrac{\vdash e \qquad \dfrac{\vdash l^{\perp}}{\vdash l^{\perp} \oplus (c^{\perp} \& d^{\perp})}}{\vdash e \otimes (l^{\perp} \oplus (c^{\perp} \& d^{\perp}))} \qquad\qquad \dfrac{\vdash e \qquad \dfrac{\dfrac{\vdash c^{\perp} \quad \vdash d^{\perp}}{\vdash c^{\perp} \& d^{\perp}}}{\vdash l^{\perp} \oplus (c^{\perp} \& d^{\perp})}}{\vdash e \otimes (l^{\perp} \oplus (c^{\perp} \& d^{\perp}))}$$

which may be written under the focused form :

$$\dfrac{e^{\perp} \vdash \qquad l \vdash}{\vdash e \otimes (l^{\perp} \oplus (c^{\perp} \& d^{\perp}))} \qquad\qquad \dfrac{e^{\perp} \vdash \qquad \dfrac{\vdash c^{\perp} \quad \vdash d^{\perp}}{c \oplus d \vdash}}{\vdash e \otimes (l^{\perp} \oplus (c^{\perp} \& d^{\perp}))}$$

In Ludics, they correspond to the following designs :

$$\dfrac{\xi.1 \vdash \qquad \xi.2 \vdash}{\vdash \xi} \qquad\qquad \dfrac{\xi.1 \vdash \qquad \dfrac{\vdash \xi.3.1 \quad \vdash \xi.3.2}{\xi.3 \vdash}}{\vdash \xi}$$

Bob's strategies correspond to :

$$\dfrac{\dfrac{\vdash e^{\perp}, l \qquad \dfrac{\vdash e^{\perp}, c}{\vdash e^{\perp}, c \oplus d}}{\vdash e^{\perp}, l \& (c \oplus d)}}{\vdash e^{\perp} \wp l \& (c \oplus d)} \qquad\qquad \dfrac{\dfrac{\vdash e^{\perp}, l \qquad \dfrac{\vdash e^{\perp}, d}{\vdash e^{\perp}, c \oplus d}}{\vdash e^{\perp}, l \& (c \oplus d)}}{\vdash e^{\perp} \wp l \& (c \oplus d)}$$

and under a focused format :

$$\dfrac{\vdash e^{\perp}, l \qquad \dfrac{c^{\perp} \vdash e^{\perp}}{\vdash e^{\perp}, c \oplus d}}{e \otimes (l^{\perp} \oplus (c^{\perp} \& d^{\perp})) \vdash} \qquad\qquad \dfrac{\vdash e^{\perp}, l \qquad \dfrac{d^{\perp} \vdash e^{\perp}}{\vdash e^{\perp}, c \oplus d}}{e \otimes (l^{\perp} \oplus (c^{\perp} \& d^{\perp})) \vdash}$$

In Ludics :

$$\cfrac{\vdash \xi 1, \xi 2 \qquad \cfrac{\xi.3.1 \vdash \xi 1}{\vdash \xi.1, \xi.3}}{\xi \vdash} \qquad\qquad \cfrac{\vdash \xi 1, \xi 2 \qquad \cfrac{\xi.3.2 \vdash \xi 1}{\vdash \xi.1, \xi.3}}{\xi \vdash}$$

Each of Alice's design "normalises" with each of Bob's. For the time being, that means simply that their strategies interact in a coherent way : each time Alice (resp. Bob) commits an action, Bob (resp. Alice) has a corresponding action to react in his (resp. her) directory.

3.3 Interaction and orthogonality

It remains to give a precise formulation of the notion of normalization of two designs. Girard defines interaction in the following way :

Interaction is coincidence of two dual loci in two different designs

Orthogonality is defined via the notion of *normalization*. Let us recall that in a vector-space, two subspaces are orthogonal if and only if their intersection is simply the null subspace $\{0\}$, and of course $\{0\}$ is self-orthogonal. In Ludics, this role must be played by a particular design : it is obviously the *daimon*. Two *paraproofs* are orthogonals if their interaction leads to the *daimon*, or in other words, if there is a consensus : the game stops because the two players agree to end it up. It is also said that the normalization of the interaction between the two objects is convergent. This notion is made more precise in what follows.

In order to explain normalization rigourously, precise definitions are needed. The reader can find some of them in the Annex (Annex 2).

Girard introduces a slight difference between designs as drawings and designs as "desseins" (plans, projects...). In French, it is almost the same word : *dessin/dessein* and it is claimed that these words were not differentiated in the past. Nowadays, "dessin" means "drawings", and "dessein" means "plan", or *design*. By putting the emphasis on this second meaning, we insist upon the strategic meaning of paraproofs, but we shall keep uniformly the word *design* in order to denote our objects.

A *chronicle* (see Annex 2) is an alternate sequence of actions, which are either positive or negative. An action is represented by a triple (ϵ, ξ_i, I_i), with ϵ a polarity, ξ_i an address and I_i a set of integers. Only the last action may be the daimon.

Under this view, a *design* is simply a set of chronicles.

In this presentation, we have now objects associated with each "player", where player 1 (resp. 2) is characterised by a set of chronicles which tell him/her what to do when facing an action of player 2 (resp. 1).

For instance, in the real estate example, the following sequence of actions is a chronicle

from X's viewpoint :

$$(+, <>, \{0\}), (-, 0, \{1, 3\}), (+, 01, \{1\}), (-, 011, \{i\})\dagger$$

Since the basis is positive (X begins with a question), the first action is of polarity $+$, its address is the empty sequence (an arbitrary locus where the question is localized) and the selected set is $\{0\}$. The second action is of negative polarity, its focus coincides with the only element of $\xi_0 \star I_0$ (the only element created by the previous action 0), noted "0". It could be interpreted as "X is ready for an answer from his/her partner localized at the sub-*loci* 01 and 03". The third one is positive, its focus is an element of $\xi_1 \star I_1 = \{01, 03\}$, here : 01. The fourth one is negative, its focus is an element of $\xi_2 \star I_2 = \{011\}$, here : 011, and X is expecting an integer as an answer to his/her question. The last action is positive and is the daimon.
Other sequences can provide chronicles, like :

$$(+, <>, \{0\}), (-, 0, \{2, 3\}), \dagger$$

In the Bob and Alice example, in order to find the chronicles, we have only to reason in terms of addresses.

1. Bob begins the dialogue by proposing the contract 1 starting from the address $<>$, this move is represented by $(+, <>, \{1\})$,

2. then, according to the choice that Alice can make, he prepares himself to give the book : $(-, 1, \{1, 2\})$, or to give the "surprise" : $(-, 1, \{1, 3\})$,

3. if Alice has chosen the book, Bob will perform a positive action : $(+, 12, \{1\})$,

4. if Alice has chosen the surprise, he will perform the positive action $(+, 13, \{1\})$ or the positive action $(+, 13, \{2\})$

hence, from Bob's viewpoint, the following chronicles :
$(+, <>, \{1\}), (-, 1, \{1, 2\}), (+, 12, \{1\})$
$(+, <>, \{1\}), (-, 1, \{1, 3\}), (+, 13, \{1\})$
$(+, <>, \{1\}), (-, 1, \{1, 3\}), (+, 13, \{2\})$
and from Alice's viewpoint :
$(-, <>, \{1\}), (+, 1, \{1, 2\}), (-, 12, \{1\}), (+, \dagger)$
$(-, <>, \{1\}), (+, 1, \{1, 3\}), (-, 13, \{1\}), (+, \dagger)$
$(-, <>, \{1\}), (+, 1, \{1, 3\}), (-, 13, \{2\}), (+, \dagger)$
This definition of chronicles and of designs as sets of chronicles allows to understand why a play, identified with a chronicle, is not simply a sequence of rule applications : if it was the case, we should expect the negative moves associated not with sets I, as it is the case,

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but with sets of sets.

Let us now define more precisely the notion of interaction (and orthogonality). As we mentioned above, A **cut** is the coincidence of two loci in the bases of two designs. A cut between $\vdash \xi$ et $\xi \vdash$ is often met, in that case, we will speak of the set made of both designs interacting by means of this cut as a "closed net". Namely, a **closed net** consists in a cut between the two following designs :

$$
\begin{array}{cc}
\mathcal{D} & \mathcal{E} \\
\vdots & \vdots \\
\dfrac{\quad}{\vdash \xi}\,\kappa & \dfrac{\quad}{\xi \vdash}\,(\xi, \mathcal{N})
\end{array}
$$

The normalization of such a closed net is such that :

 – if κ is the daimon, then the normalized form is :

$$
\dfrac{\quad}{\vdash}\,\dagger
$$

   (this normalised net is called dai)

 – if $\kappa = (\xi, I)$, then if $I \notin \mathcal{N}$, normalization fails,

 – if $\kappa = (\xi, I)$ and $I \in \mathcal{N}$, then we consider, for all $i \in I$ the design $\mathcal{D}_i$, sub-design of $\mathcal{D}$ of basis $\xi \star i \vdash$, and the sub-design $\mathcal{E}'$ of $\mathcal{E}$, of basis $\vdash \xi \star I$, and we replace $\mathcal{D}$ and $\mathcal{E}$ by, respectively, the sequences of $\mathcal{D}_i$ and $\mathcal{E}'$.

In other words, the initial net is replaced by :

$$
\begin{array}{ccc}
\mathcal{D}_{i_1} & \mathcal{E}' & \mathcal{D}_{i_n} \\
\vdots & \vdots & \vdots \\
\dfrac{\quad}{\xi \star i_1 \vdash}\quad \ldots\quad \dfrac{\quad}{\vdash \xi \star i_1, ..., \xi \star i_n} & & \dfrac{\quad}{\xi \star i_n \vdash}
\end{array}
$$

with a cut between each $\xi \star i_j \vdash$ and the corresponding "formula" $\xi \star i_j$ in the design $\mathcal{E}'$

This definition formalizes the intuition we mentioned above, of an interaction which ends up by a consensus (or, to the contrary, which diverges, like it is the case when $I \notin \mathcal{N}$).

Moreover, orthogonality may be now well defined :

A design $\mathcal{D}$ is orthogonal to a design $\mathcal{E}$ ($\mathcal{D} \perp \mathcal{E}$) if and only if the elimination of the cut between the two leads to the net dai. Let us look how to prove orthogonality of two designs,

by taking again the real estate example. Let us consider the two following designs, linked by a cut at their basis :

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{\rule{2cm}{0.4pt}}{... \vdash 011i, 02...}\;\dagger
      }{011 \vdash 02}\;(011, \{i; i \in \mathbb{N}\})
    }{\vdash 01, 02}
  }{0 \vdash}
}{\vdash <>}
\qquad
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{011i \vdash}{\vdash 011}
    }{01 \vdash \quad 02 \vdash}
  }{\vdash 0}
}{<> \vdash}
$$

It is possible, in several stages, to make the cut go up. For instance, a first stage leads to :

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{\rule{2cm}{0.4pt}}{... \vdash 011i, 02...}\;\dagger
    }{011 \vdash 02}\;(011, \{i; i \in \mathbb{N}\})
  }{\vdash 01, 02}
}{0 \vdash}
\qquad
\cfrac{
  \cfrac{
    \cfrac{011i \vdash}{\vdash 011}
  }{01 \vdash \quad 02 \vdash}
}{\vdash 0}
$$

a second stage to :

$$
\cfrac{
  \cfrac{011i \vdash}{\vdash 011}
}{01 \vdash}
\qquad
\cfrac{
  \cfrac{
    \cfrac{\rule{2cm}{0.4pt}}{... \vdash 011i, 02...}\;\dagger
  }{011 \vdash 02}\;(011, \{i; i \in \mathbb{N}\})
}{\vdash 01, 02}
\qquad\qquad 02 \vdash
$$

where X's design has been splitted in two parts, for readibility, and where there is a cut between $01 \vdash$ and the 01 which is in $\vdash 01, 02$ on one side, and another one between the 02 there is in $\vdash 01, 02$ and $02 \vdash$ on the other side.
A third stage leads to :

$$
\cfrac{011i \vdash}{\vdash 011}
\qquad
\cfrac{
  \cfrac{\rule{2cm}{0.4pt}}{... \vdash 011i, 02...}\;\dagger
}{011 \vdash 02}\;(011, \{i; i \in \mathbb{N}\})
\qquad\qquad 02 \vdash
$$

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then we get :

$$
011i \vdash \quad \overline{\vdash 011i, 02}^{\,\dagger} \quad 02 \vdash
$$

with a cut between $011i \vdash$ and the $011i$ which occurs in the sequence of sequents $\{\vdash 011i, 02; i \in \mathbb{N}\}$,, and between the 02 and $02 \vdash$. Because each cut possesses a member obtained by the daimon, the interaction converges towards dai.

3.4 Separation

Designs are like strategies. As a metaphor, we can see them as a set of potential plays in which a determinate player may makes his/her choice, but how to use a design $\mathcal{D}$ of base $\vdash \sigma$? As Girard says : "simply by cutting it with a *counter-design* $\mathcal{E}$ of base $\sigma \vdash$ and normalizing the net they constitute". What is very important here is that designs are determined by their orthogonal, that means *their use*. Hence the importance of the *Separation Theorem* :

> If $\mathcal{D} \neq \mathcal{D}'$ then there exists a counterdesign $\mathcal{E}$ which is orthogonal to one of $\mathcal{D}, \mathcal{D}'$ but not the other.

Otherwise, in order to play at best, the player needs to organize his/her designs according to their possible uses. We therefore define a partial order on designs, made possible by the separation theorem which guarantees that the relation is indeed an order.

> $\mathcal{D} \preceq \mathcal{D}'$ if and only if every design orthogonal to $\mathcal{D}$ is also orthogonal to $\mathcal{D}'$ ($\mathcal{D}^{\perp} \subset \mathcal{D}'^{\perp}$)

In *From foundations to Ludics* ([10]), Girard makes this definition more intuitive : $\mathcal{D} \preceq \mathcal{D}'$ means that $\mathcal{D}'$ is "more convergent" than $\mathcal{D}$. We may think of $\mathcal{D}'$ as obtained from $\mathcal{D}$ by means of two operations :

- *Enlargement* : we add more premises for positive rules, in other words we replace $\mathcal{N}$ by $\mathcal{N}' \supset \mathcal{N}$ (in other words, less branches of the dialogue will lead to a dissensus),
- *Shortening* : some positive rules (ξ, I) are replaced by simple occurrences of the *daimon* (on some branches, the dialogue ends up earlier).

This definition gives sense to the notions of a smallest and a biggest design. The smallest one is not a design properly speaking, it conducts the most often to dissensus, it has no branches and it is the empty set of chronicles. Girard calls it *Fid*. The biggest one always converges and at once, it is the *daimon*.

We may distinguish a positive *daimon* and a negative one.
Positive daimon :

$$\overline{}_{\vdash \Lambda} \dagger$$

Negative daimon :

$$\cfrac{... \quad \cfrac{\overline{}}{\vdash \xi \star I, \Lambda} \dagger \quad ...}{\xi \vdash \Lambda} (\xi, \wp_f(\mathbb{N}) \setminus \{\emptyset\})$$

If we take $\mathcal{N} = \emptyset$ in the positive rule, then obviously, for any move made by the other player, there will be a dissensus (the proponent has not enough information to answer to the attacks of the opponent). This case is represented by the *skunk* :

$$\overline{}_{\xi \vdash} (\xi, \emptyset)$$

Looking at the real estate example, we see that the design of X ($\mathcal{D}_X$) is orthogonal to several possible designs of Y, its orthogonal therefore contains all these designs. Let us now imagine another design, $\mathcal{D}'_X$ which is orthogonal to still more designs, in other words, where X would have planned still more moves by his or her partner. This design would contain more chronicles ended up by the daimon, and chronicles that would develop starting from other negative actions and also leading to *daimon*.
If we consider now the orthogonal of $\mathcal{D}^\perp$, then obviously it contains $\mathcal{D}$, but it also contains all the designs $\mathcal{D}'$ such that $\mathcal{D} \preceq \mathcal{D}'$. In other words, it is stable by super-design.

4   Behaviors

When we have a non empty set of designs, we are interested in all the ways of extending it by means of other ones which would behave similarly with regards to normalization. The completed set is called a *behavior* : we may actually think of it as the extension of a real behaviour (in the sense of a set of coherent primitive actions, which intends to reach some goal).

Définition 1 *A* **behavior** *is a set of designs (of the same basis)* **C** *which is equal to its bi-orthogonal (* **C** *=* **C**$^{\perp\perp}$*).*

Given a design $\mathcal{D}$ belonging to a behavior **C**, there exists in **C** a smallest element included in $\mathcal{D}$ which is, in some way, its best witness, we call it the *incarnation of $\mathcal{D}$ with regards to* **C**. It is the smallest part of a design which guarantees its belongness to a behavior. This notion reveals useful when looking for a new interpretation of connectives in Ludics terms.

4.1 Some particular behaviors

We may now define the behavior induced by a design $\mathcal{D}$, it is the bi-orthogonal of the set $\{\mathcal{D}\}$. In the example we have been dealing with until now, it contains all the designs of $\mathcal{D}_X$, and particularly the *daimon* of basis $\vdash <>$, design which did not occur in our example as a design of X, but that could be now chosen : it would correspond to an immediate issue from the dialogue, even before having asked the initial question.

Another example is given by the following design :

$$\frac{}{\vdash \xi} \, (\xi, \emptyset)$$

It is very particular. it is a positive design, therefore held by the proponent. If (s)he aims at a consensus, the opponent has not many choices since there is no address for answering (the proponent gave none). The opponent can therefore only oppose the *daimon*, but in doing so, (s)he admits that (s)he has lost the game ! With such a design... the proponent ever wins ! Girard therefore calls it the *Nuclear Bomb*, because it is the Absolute weapon ! Its only orthogonal design is therefore :

$$\frac{\dfrac{}{\vdash}^{\dagger}}{\xi \vdash}$$

but this last design is also orthogonal to :

$$\frac{}{\vdash \xi}^{\dagger}$$

Therefore we have a behavior which contains *two* designs : the *Bomb* and the *daimon*. let us note **1** this behavior : it will be the neutral element of the tensor.

But instead of the *Bomb*, we may take the *skunk* and look for the negative behavior which contains it. It will be noted $\top$. What is the orthogonal of $\top$? Since the other player does not give me any set of addresses to play, I can only play the *daimon*, in other words I loose. We get :

$$\top^{\perp} = \{Dai\}$$

and

$$\top = Dai^{\perp}$$

that is, all the negative designs of the same basis.

4.2 What Logic for Ludics ?

4.2.1 "Truth" and "falsehood"

Having a notion of *win* (a design is said to be *winning* if it does not play the *daimon* and it is parcimonious and uniform) allows us to get back a notion of "truth" : a behavior could be said "true" when it contains a winning design, false when its orthogonal contains one. With this characterization, two designs which incarnate two orthogonal behaviors cannot be both true. Or else, if they are both winning, they are not orthogonal, that means that *their "dispute" goes on indefinitely*. But we can have behaviors which are neither true nor false, it is the case if the two players give up.
Probably this attribution of "truth" to a behavior is not particularly adequat... Perhaps it could be better to speak of a "fruitful" behavior... But this is only to show that even in such an anti-realist conception, it is still possible to introduce a duality which can be compared to "true vs false".

4.2.2 Formulae and behaviors

We may notice here that behaviors are sets of designs exactly like in intuitionistic logics, formulae are interpreted as sets of proofs. This leads to the following correspondance :

formula	behavior
proof	design

When we speak of the intersection of two behaviors, it is as if we spoke of the intersection of two formulae. In intuitionistic logic, this has an obvious meaning : an element of this set proves A and proves B and therefore it proves A∧B, but in Ludics, things go differently. The notion of formula as a "spiritualist" notion evaporates and is replaced by that of locus. There are therefore two ways of speaking of intersection and union, one takes into account this concrete aspect of the formulae (their localisation) and the other one takes into account the possibility we have to "de-localise" them (particularly by means of *Fax*).

4.2.3 A localist viewpoint against a spiritualist one

It is clear that in Set Theory, the union is *localist* : it crucially depends on the elements of the sets. If we take two isomorphisms ϕ and ψ, we shall not have in general $X \cup Y = \phi(X) \cup \psi(Y)$. For instance, it is not enough to know the cardinality of X and that of Y to know that of $X \cup Y$.
But it is possible to make the disjoint union of two sets X and Y. For that we shall have to take two one-to-one mappings ϕ and ψ such that $\phi(X)$ and $\psi(Y)$ are disjoint and we shall

have of course : $Card(X + Y) = Card(X) + Card(Y)$. The disjoint sum is therefore *spiritualist*, that means : it does not depend on concrete localisation.

In Ludics, it is possible to perform some operations like in the case of the disjoint union. If a locus σ is given, we may define two applications ϕ et ψ which will be seen as *delocalisations* :

$$\phi(\sigma \star i \star \tau) = \sigma \star 2i \star \tau \qquad \psi(\sigma \star i \star \tau) = \sigma \star (2i + 1) \star \tau$$

The images by ϕ and ψ are still designs. If **G** and **H** are behaviors of the same basis σ, then $\phi(\mathbf{G})$ and $\psi(\mathbf{H})$ are disjoint. Or, more exactly, when they are positive behaviors, their intersection is the smallest behavior on a positive basis, that is $\mathbf{0} = \{Dai\}$, and when they are negative behaviors, every design $\mathcal{D}$ which belongs to both is such that the intersection between its incarnation with regards to one and its incarnation with regards to the other is empty.

4.3 Sums and products

If A is an arbitrary set of designs, then $A^\perp$ is a behavior (since $A^{\perp\perp\perp} = A^\perp$). Reciprocally, every behavior on a basis $<>\vdash$ or on a basis $\vdash<>$ is the orthogonal of a set of designs. For instance, if $A = \emptyset$, $A^\perp$ is the set of designs which converge with no other one, that is which converge only with Dai. It is $\top$, the behavior that the *skunk* is the incarnation of. Otherwise, since behaviors are sets, nothing prevents to make unions, intersections and products of them. It is possible to show :

   – *The intersection of two behaviors is a behavior*

We may also define a behavior from a union. Let us write :

$$\mathbf{G} \sqcup \mathbf{H} = (\mathbf{G} \cup \mathbf{H})^{\perp\perp}$$

this will allow us to retrieve the connectives of Linear Logic.

4.3.1 Additives

The case of additives corresponds to the case where behaviors are disjoint.

Définition 2 *A* **directory** *is a subset of* $\wp_f(\mathbb{N})$ *(= a set of "ramifications"). If* **G** *is a positive behavior on* $\vdash<>$*, then the directory of* **G***, noted* $Dir(\mathbf{G})$ *is the set of the sets of indexes I such that :*
$(+, <>, I)$ *that is the first action of a design belonging to* **G***.*
If **G** *is a negative behavior on* $\xi \vdash$*,then the directory of* **G** *is the set of sets of indexes I for which, for any design incarnated in* **G***, there is an action* $(-, \xi, I)$ *as the initial action,*

This definition entails (P. L. Curien p.43) that the directory of **G** and that of **G**$^\perp$ are identical.

Définition 3 *Two behaviors* **G** *and* **G'** *on the same basis are said to be disjoint if their directories are disjoint.*

If two negative behaviors **G** and **H** are disjoint and if $\mathcal{D}_1$ and $\mathcal{D}_2$ are incarnations of these behaviors, then their union is well defined, and it is a design of **G** $\cap$ **H**. Its first action has, as its set of ramifications, the union of that of $\mathcal{D}_1$ and of that of $\mathcal{D}_2$, it is therefore a design which is inside **G** and inside **H**, but it is at the same time, a pair of designs, thus leading to :

$$|\mathbf{G}| \times |\mathbf{H}| = |\mathbf{G} \cap \mathbf{H}|$$

where $|\mathbf{G}|$ denotes the set of incarnations with regards to **G**. In other words, every incarnated design of **G**$\cap$**H** is obtained by taking a design incarnated in **G** and a design incarnated in **H** making the union of the chronicles. It is what Girard calls the mystery of incarnation (the fact that intersection and cartesian product coincide).
It is then possible to define the additive connectives.

Définition 4 *If* **G** *and* **H** *are two disjoint negative behaviors, then, we put :*

$$\mathbf{G}\&\mathbf{H} = \mathbf{G} \cap \mathbf{H}$$

If they are positive, we put :
$$\mathbf{G} \oplus \mathbf{H} = \mathbf{G} \sqcup \mathbf{H}$$

$\oplus$ is therefore a spiritualist connective.

4.3.2 Multiplicatives

Définition 5 *Let $\mathcal{U}$ and $\mathcal{B}$ be two positive designs, we define the tensor product $\mathcal{U} \odot \mathcal{B}$ by :*
   – *if one of the two is the daimon, then $\mathcal{U} \odot \mathcal{B} = Dai$,*
   – *if not, let $(+, <>, I)$ and $(+, <>, J)$ be the first actions of respectively $\mathcal{B}$ and $\mathcal{U}$, if $I \cap J \neq \emptyset$, then $\mathcal{U} \odot \mathcal{B} = Dai$. If not, we replace in each chronicle of $\mathcal{B}$ and $\mathcal{U}$ the first action by $(+, <>, I \cup J)$, which gives respectively $\mathcal{B}'$ and $\mathcal{U}'$, then $\mathcal{U} \odot \mathcal{B} = \mathcal{U}' \cup \mathcal{B}'$.*

It is then possible to define the tensor product $\otimes$ of two behaviors by means of delocalisations, but notice that Ludics also provides us with new connectives, with regards to Linear Logic. Theses de-localisations (for instance the previous ϕ and ψ which send biases following the locus of the basis either to odd integers or to even ones) allow to make that the first actions respective to the two designs which incarnate the two behaviors have disjoint

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ramifications. We may define the product $\odot$ of two behaviors by : $\mathbf{F} \odot \mathbf{G} = \{\mathcal{A} \odot \mathcal{B}; \mathcal{A} \in \mathbf{F}, \mathcal{B} \in \mathbf{G}\}$, and then the tensor properly speaking by $\mathbf{F} \otimes \mathbf{G} = (\mathbf{F} \odot \mathbf{G})^{\perp\perp}$.

Under full completeness (that is under the assumption that the bi-orthogonal does not contain more than the initial behavior), we shall have that every design $\mathcal{D}$ in $\mathbf{F} \otimes \mathbf{G}$ may be decomposed according to : $\mathcal{D} = \mathcal{D}_1 \odot \mathcal{D}_2$ with $\mathcal{D}_1 \in \mathbf{F}$ and $\mathcal{D}_2 \in \mathbf{G}$.

Nevertheless, let us go back to our new connective by means of a striking example where we see that the locative viewpoint enables one to take into account some "non logical phenomena", like the possibility to obtain $true \odot true = false$. Let $\mathbf{G} = \{\mathcal{D}\}^{\perp\perp}$, where $\mathcal{D}$ is the design :

$$\cfrac{\cfrac{}{0 \vdash} \; (0, \emptyset)}{\vdash <>} \; (<>, \{0\})$$

in other words : I ask the question 0, then I don't open any possible answer for my opponent, therefore of course, I win ! Let us calculate $\mathbf{G} \odot \mathbf{G}$. Since $I \cap J \neq \emptyset$, the definition gives us : $\mathbf{G} \odot \mathbf{G} = Dai$, it is the behavior which contains the only design :

$$\cfrac{}{\vdash <>} \; \dagger$$

It is the behavior which immediately looses. Therefore we may very well have :

$$true \odot true = false$$

We can have the following interpretation. Let us imagine $\mathbf{G}$ to be the behavior generated by a design $\mathcal{D}$ which consists in the possibility to assert 0 from a situation localized in $<>$. Let $\mathbf{H}$ the behavior generated by the following $\mathcal{E}$

$$\cfrac{... \; \cfrac{}{\vdash n} \; \dagger \quad ...}{<> \vdash} \; (-, <>, \{\{n\}; n \in \mathbb{N}\})$$

which can be interpreted as : in the situation $<\;>$ as soon as you give me an argument, I shall give up.

Of course, $\mathcal{D}$ and $\mathcal{E}$ are orthogonal, therefore $\mathcal{D} \in \mathbf{H}^\perp$, something that we translate by saying that $\mathbf{G}$ wins against $\mathbf{H}$. Moreover, $\mathbf{G}$ is *true* since it has a winning design. But $\mathbf{G}$ does not contain only $\mathcal{D}$, it also contains de-localized designs of $\mathcal{D}$. For instance, we have seen that the argument given in $\mathcal{D}$ was 0, we can now imagine a variant with the argument 1. Let $\mathcal{D}'$ this variant, it is a design of basis $<> \vdash$ and its only chronicle is : $(<>, \{1\})$. Let

us compare $\mathcal{D} \odot \mathcal{D}$ and $\mathcal{D} \odot \mathcal{D}'$. $\mathcal{D} \odot \mathcal{D}'$ is no longer orthogonal to $\mathcal{E}$, therefore it no longer belongs to $\mathbf{H}^{\perp}$. It is now orthogonal to what we get with another design $\mathcal{E}'$ of the kind :

$$\frac{\overline{} \quad \overline{\vdash n, n'}^{\,\dagger} \quad \overline{}}{<>\vdash} \; (\{\{n, n'\}; n, n' \in \mathbb{N}, n \neq n'\})$$

which now interprets as "give me two arguments, and I shall give up". Because $\mathcal{D} \odot \mathcal{D}'$ is orthogonal to $\mathcal{E}'$, $\mathcal{D} \odot \mathcal{D}'$ belongs to the orthogonal of the behavior $\mathbf{H}'$ generated by $\mathcal{E}'$. But let us remember that $\mathcal{D} \odot \mathcal{D} = Dai$, and $\mathbf{G} \odot \mathbf{G} = \{Dai\}$. It follows that $\mathcal{D} \odot \mathcal{D}$ cannot be winning. If I have only one argument to give, I cannot give two... Hence the fact that if $\mathbf{G}$ is winning against *give me an argument*, in order for $\mathbf{G} \odot \mathbf{G}$ to be true, it should converge with a $\mathbf{H'}$ which would be interpreted as *give me two arguments*, which is not the case.

The tensor product thus expresses, like we expected, a resource sensitivity. On the viewpoint of a "perfect dialogue", each question calls for one answer and an answer does not satisfy two questions at the same time, or there must be two instances (situated in two different *loci*) of the answer, each answering to a separate question.

5 Conclusion

In the previous sections, we gave some definitions and examples of the main concepts of Ludics. Definitions were not always precise and formal ones, but we hope that by means of them, the reader got the essence of these notions. Let us now try to express the core philosophical ideas contained in that frame. A. Pietarinen already exposed some of them in a very relevant way, comparing the spirit of Ludics with Peirce's pragmatics and Wittgenstein's philosophy ([16]). Making comments on the fact that in Ludics, the meaning of a formula (or the meaning of the proof of a formula) is given by its interaction against observers ("Proofs, and hence meanings, are created solely by interaction with other proofs"), the Finnish philosopher points out the commonality of this kind of operational meaning with that of Peirce's pragmatic maxim, according to which the meaning of a proposition does not lie in its "conformity" to an exterior state of affairs, but is provided by another sign, its *interpretant* (the interpretant is "a sign of another sign"). In Ludics, this translates into the fact that the meaning of a proof is given by the proofs themselves, and paraproofs may be considered as producing interpretants for proofs.

Pietarinen (p. 273) also points out that these ideas may be paralleled with Wittgenstein's remark : "you can't get behind rules, because there isn't any behind" (Wittgenstein, 1978, p. 244).

Immanentism is manifest in Ludics when compared to the traditional Theory of Games.

For instance, Ludics ignores *pay-off* functions which would determine the "winner" from the outside. Instead of that, it relies solely on the internal structure of the interactions. Like Pietarinen says "the internal process of computation is thus taken to be as interesting as the outcome". We may think that this really paves the way for a general theory of games, more in conformity with Wittgenstein's language games than current semantical games, since of course the language games that Wittgenstein envisaged, constitued a much more complicated network than given by usual logical games which can be put in an "extensive form".

Nevertheless, as we already noticed, the notions of a game and of a strategy are, in Ludics, a mere metaphor. Ludics can get rid of them. There are actually no real players like "Nature" and "Myself" having always played a game since immemorial times ! Objects which interact are not individual players but sets of designs called "behaviors". There is no room for empirical subjects, and "behaviors" are just what replaces "formulae" in older frameworks (and designs just what replaces proofs of formulae).

With regards to "anti-realism", comparison maybe made with Dummett's philosophy ([5]), it would not be right to say that Ludics denies reality, more than does intuitionnistic logic. It only starts from the observation (in common with Dummett) that relying on a correspondence theory in order to found the relation between language and reality leads to a dead end. Therefore, coherently, another conception must be found regarding our intellectual activities. That does not mean that reality does not exist but only that, if it exists, it is only as what makes processes, and interactions between them, possible.

6   Annexes

6.1   Annex 1 - A tryadic sequent calculus

In his proof of the focalization theorem, Andréoli uses sequents with a *stoop*. The stoop contains at most one formula and it is positive. This corresponds to the case where, searching for an active formula, we meet one which is positive : at this time, we focus on it and the following rules only apply to the content of the stoop, producing new positive formulae to decompose until we find a negative one, in which case, this last one is put in the common space in order to begin a sequence of negative steps. The reintegration of a formula in this common space amounts to a *shift*, that is a step where the polarity is changed. The formula which goes to the negatives was until now among the positives, and therefore seen as positive. This entails that a formula can change its polarity. In order to facilitate the calculus, it is in fact considered that every connective of a given polarity connects formulae of the same polarity. For instance, if we have a tensor between negative formulae, we shall have an operator of change of polarity which will temporarily make positive the negative

formulae. If N is negative, $\downarrow N$ is positive, and the rule for changing sign is the rule :

$$\frac{\vdash \Delta, N;}{\vdash \Delta; \downarrow N} \ [\downarrow]$$

In a similar way, the focalization process, that is the process according to which by decomposing a negative formula a positive formula is found, which is focused, may be described by means of two rules. On one side, another rule for changing the sign, which makes the formula on which we shall focus, until now among the negatives and therefore negative itself, to go back to its positive polarity, and on the other side the focalization rule properly speaking, which makes any positive formula go to the stoop. These two rules are respectively :

$$\frac{\vdash \Delta, P;}{\vdash \Delta, \uparrow P;} \ [\uparrow] \qquad\qquad \frac{\vdash \Delta; P}{\vdash \Delta, P;} \ [focus]$$

In the focalization rule, the only negative elements of Δ are necessarily atoms (if not, there would be still a negative formula to decompose). The other rules are then :

AXIOM

$$\overline{\vdash P^{\perp}; P}$$

POSITIVE RULES

$$\frac{\vdash \Gamma; P}{\vdash \Gamma; P \oplus Q} \qquad\qquad \frac{\vdash \Gamma; Q}{\vdash \Gamma; P \oplus Q} \qquad\qquad \frac{\vdash \Gamma_1; P \qquad \vdash \Gamma_2; Q}{\vdash \Gamma_1, \Gamma_2; P \otimes Q}$$

NEGATIVE RULES

$$\frac{\vdash N_1, N_2, \Gamma;}{\vdash N_1 \wp N_2, \Gamma;} \qquad\qquad\qquad \frac{\vdash N_1, \Gamma; \qquad \vdash N_2, \Gamma;}{\vdash N_1 \& N_2, \Gamma;}$$

6.2   Annex 2 - Designs and chronicles

Définition 6 *A **design** is a tree, all the nodes of which are **forks** $\Gamma \vdash \Delta$ the root of which is called a **basis** (or conclusion), and which is built by using :*
   – the daimon
   – the positive rule
   – the negative rule

On another hand, designs as "desseins" are sets of "chronicles". A *chronicle* is an alternate sequence of moves (or *actions*). More precisely :

Définition 7 *A chronicle of basis $\Gamma \vdash \Delta$ is a non empty sequence of alternate actions $k_0, ..., k_n$, where $k_i = (\epsilon, \xi_i, I_i)$, with ϵ a polarity, ξ_i an address and I_i a set of integers, such that :*
  – *if the basis is negative (resp. positive), k_0 has the polarity $-$ (resp. $+$),*
  – *only the last action, k_n may be the daimon = $(+, \dagger)$,*
  – *if we call **focus** the locus of the active formula, a negative action k_i has, as its focus, either the unique element of Γ (in this case, the first action), or an element of $\xi_{i-1} \star I_{i-1}$,*
  – *the focus of a positive action k_i is either an element ξ_i of Δ or an element of $\xi_q \star I_q$, where $(-, \xi_q, I_q)$ is a previous negative action,*
  – *the focuses are pairwise distinct.*

We may make more precise the meaning of design :

Définition 8 *A **design** of basis $\Gamma \vdash \Delta$ is a set $\mathcal{D}$ of basic chronicles $\Gamma \vdash \Delta$ such that :*
  – *$\mathcal{D}$ is a forest,*
  – *the chronicles of $\mathcal{D}$ are pairwise coherent (if two chronicles differ, it is only at a negative action, and after it, they have never the same foci),*
  – *if a chronicle ends up, its last action is positive,*
  – *if the basis is positive, then $\mathcal{D}$ is non empty.*

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